Introduction

Thinking the Parthenon and Liberal Arts Education Together

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This work is simultaneously expansive and narrow in its scope. It is expansive in that it takes up three objects that would generally be thought to diverge widely: (1) the early history of Greek mathematics; (2) Plato's *Republic* and *Timaeus*, in conversation with the program of study in the early Academy; and (3) the Parthenon. It is narrow in that it investigates each of these objects with respect to Philolaus's interdisciplinary research into number theory, astronomy, and harmonics. This narrow thematic focus allows us to say something quite distinct about each of our objects, and then to relate them to one another. In this introduction, we discuss possible connections between sixth- and seventh-century BCE Greek mathematics and earlier Near-Eastern predecessors to show the possible origins of Philolaus's insights in these fields. In part I, we investigate Plato's reception of Philolaus's work to understand how dialectic and mathematics function both in *Republic* and *Timaeus* and in the intellectual environment of the early Academy. In part II, we will present our reading of the Parthenon as a "vanishing mediator"[1] between the earliest developments in Greek mathematics and the sophisticated extension of those early developments in Plato's dialogues and the early Academy. Specifically, we will be concerned to show how the themes of Philolaus's work (fl. ca. 440–410 BCE), roughly contemporary with the construction of the Parthenon, embodied in *symmetria* (commensurability) and *harmonia* (harmony; joining together), relate to the design features of the Parthenon (447–432 BCE) as they make manifest the theological (ontological) and civic (educational) meaning of the building. A brief afterword will advance an understanding of the relationship between humanist learning and technical achievement through procedural knowledge that we believe shows how one might see a continuous development from the earliest advances of Greek mathematics through fifth-century developments such as Philolaus and the Parthenon through to Plato's Academy, looking also at the analogous situation in the Renaissance.

1. The Parthenon as an Institution of Liberal Arts Education

We propose here to pursue a method of speculative reconstruction to detail what can be learned about the "state of the art" in the early development of "liberal education" in fifth-century Greece. One needs to be cautious in speaking about such a development at such a time, which predates the establishment of any independently operating institution that might naturally be thought to pursue such an educational project in today's terms. The Parthenon, the foremost example of the practical application of mathematical knowledge in the mid-fifth century, insofar as it displays the cultural milieu in which mathematical knowledge was growing in both sophistication and in audience at the relevant time, can be understood as such an institution for liberal arts education. Specifically, coming to see the Parthenon as a manifestation in material form of the quest to achieve a formal integration of the mathematical arts points to a way in which liberal education has been, and could now be, a vital part of the civic life of a democratic society. The Parthenon is both the work of a well-educated group of theoretician-practitioners of mathematical knowledge and a work for the cultivation of a certain kind of generally educated citizen. Understood in this light, it helps us see the roots of the *trivium* and *quadrivium*[2] as they later came to be classically conceived.

Before we attempt the comparative analysis of the Parthenon's design features as a mediator between the earliest, scarcely documented sources of Greek mathematics and the liberal arts curriculum in the Academy, we ought perhaps to say a word about why its status as such a mediator did in fact vanish. That is, if the elements that emerge from our reading of the Parthenon are really there, why don't the innovations in the Parthenon produce an explicit and immediate textual response? We feel this dilemma relates directly to the nature of the building as art object and as sacred space. As we will discuss with specific reference to elements we focus on later, the building does "theoretical work" in ways specific to the experience of a work of art, ways that need not, and ultimately cannot, be fully articulated in words. In that sense, we would not expect to find contemporaneous textual discussion of the theoretical work that the building is doing. Rather, as we see in Plato's dialogues, analogous theoretical problems emerge in later texts, not so much through direct "influence" but through broader and more indirect connections that arise from their shared intellectual culture. This is similar to the situation in Renaissance Europe, when perspective pictures, made for a sacred context, involve specifically pictorial theological interpretations (i.e., not merely duplicating what texts can do), and also implicate epistemological paradigms that would emerge in more fully theorized and elaborated forms in later centuries (e.g., in Cartesian epistemology).

With so much said for the basic orientation we bring to the Parthenon, let us now reflect briefly on the history of the liberal arts as Plato came to give them determinate form. "Pythagoras introduced the *quadrivium* to Greece." This traditional understanding—this "creation myth"—of Pythagoras as the first philosopher is attested very early in the classical canon.[3] Indeed, though the text is very understated, Plato's observation in Book 10 of the *Republic*[4] that Pythagoras, like Homer, was hailed as a "master of education" seems to point to an already-established view that holds Pythagoras as a model of what Aristotle already refers to as "liberal education."[5] Our suggestion is that the procedure we will follow in our analysis—testing a formal understanding (here: the mathematical theme of reconciling arithmetic and geometry through harmonics) against a material object (here: the Parthenon itself in its architectural and sculptural program)—is precisely the model that Pythagoras introduced as a "model educator," and the one that inspired the design of the Parthenon.

To cite just one (especially illuminating) example: if we look at the dimensions of the Parthenon's stylobate, we see that they were quite likely determined by a method, standard for Doric architecture in the first half of the fifth century, based on intercolumniations five times the width of the triglyph, that is, on a 5:1 (80:16) ratio of intercolumniation to triglyph. There is an important difference, however, in the case of the Parthenon: the continuous proportion from which the façades and flanks of the building were constructed gives a 81:16 ratio between these elements, as two elements in a continuous proportion of intercolumniation to lower column diameter to triglyph width, where the full expression is 81:36:16 (this is a *continuous* proportion since 81 and 36 are in the same ratio as 36 and 16).[6] The refinements involved with fitting these two slightly different constructive principles together—that is, these two different forms of *symmetria* (commensurability)—is a first instance, among many others we will investigate in detail, where we encounter the problem of *harmonia* (harmony, i.e., "joining together") in the Parthenon.

One important consequence of the repeated use of this particular continuous proportion, at various scales, is that it allowed for the building's overall cubic proportions, in the continuous proportion of 81 (length) : 36 (width) : 16 (height) to have as its unit the real, visible triglyph module.[7] Focusing for the moment only on the two most remarkable features of this innovation—adapting the 5:1 (80:16) ratio to the mathematically much more interesting ratio of 81:16 and the even more remarkable offering of the visible triglyph module as the unit for the building as a whole—two points emerge. First, the remarkable sophistication of the theoretical reflection at work in the monument's design becomes in a literal sense visible. Still more strikingly, we believe, the designers made this sophistication accessible to the temple's audience, which is ultimately the whole city, by weaving these formal features into our sensory and embodied experience of the building.

Through this careful consideration of the educational program behind the design of the Parthenon and in its role as a form of civic education, we hope to show that the practical arts played a key part in the birth of liberal arts education. The well-rounded education that came to be programmatic in the Academy has as its proximal antecedent the practical, but not merely practical, education in the arts that the planners of the Parthenon brought to bear in and through its construction. More than anything else, this antecedence manifests itself in the elegant interrelation of the soon-to-be-canonized mathematical arts of arithmetic, geometry, astronomy, and harmonics in the building's constructive program.[8]

Both to shed light on the notion that the Parthenon is a vanishing mediator in this sense, and by way of concluding this statement concerning the significance of our project at large, we would like to address three fundamental criticisms to which our entire method of speculative reconstruction can reasonably be subjected.

First, a more hard-headed historian might object that even if we can "read" the Parthenon as it stands as being the site of the "integrated mathematical arts" as they are canonized in the fourth century, this does not, in light of the total absence of other primary source documentation, give us reason to be certain that any significant portion of the people who designed and built the temple had any awareness of the presence of these features or the capacity to appreciate them. Even less, the criticism could continue, do we have grounds to believe such features to be among the principles of its organization. In response, we would point to the intensity of the reflective awareness the design program displays, also in comparison with Doric temple design before and after the Parthenon. This suggests that the problems we will discuss in detail below were on the minds of those responsible for the building, and are not a projection back onto it. If that is possible, then we hope to show it is also plausible that some significant portion of the "knowledge workers" assigned to this commission had a reasonably advanced understanding of principles not yet recorded in the works of theoretical mathematics from the mid-fifth century. We also aim to show that the Parthenon as designed intends for its audience—or at least a considerable part (the "educated" or cultivated part, those versed in *mousikē*, the works of the muses) of that audience—first, to recognize the presence of these problems, and then, having recognized them, to "educate themselves" in a manner not dissimilar to how Socrates defines dialectic as the "art of turning around the whole soul" (*Resp.*, 7.518d).

But, our interlocutor might insist, would it really have been the case that any number of people involved either in the design and the construction of the temple, or in visiting and making use of the space once built, would have had any access to, or interest in, the features on which we focus here? To this we would reply: the character of the Parthenon as a *work of art*, and not a theoretical written text, is crucial (and we mean "work of art" here in the broadest sense, not "art for art's sake" but an idea of art inclusive of the building's religious meaning and function). The building creates an encounter with every receptive viewer, whatever his or her educational background or degree of specialized knowledge, an encounter that is ultimately irreducible to an entirely verbal, or entirely mathematical, articulation. That encounter is first and foremost an embodied and sensory one, within which the mathematical and ontological questions the building raises are embedded, but it is never fully reducible to those questions. The analogy with music may be helpful here: in music, one can experience harmony, and have an emotional response to it, without understanding the mathematical principles involved; likewise, those mathematical principles themselves are not adequate to explain the ineffable character of music, even if they are its foundation. Thus, we can imagine that one viewer may experience *symmetria* and *harmonia* in a strictly intuitive way when encountering the Parthenon (*symmetria* in the well-ordered and pleasing proportions of the design, *harmonia* in the sense of a complex of parts holding together as one thing and in the beauty of the whole); another may connect those experiences to the religious and/or civic significance of the monument; another may speculate on the building's mathematical character and even be inspired to count and measure; and another may consider the relationship between arithmetic and geometry, reflect on the philosophical question of *harmonia*, and be led toward dialectical thought.

This last group would probably be a small number of people, and the first group would probably be the largest. Also, many of those involved in making the Parthenon may have had specific technical knowledge that need not have involved awareness of all the larger philosophical questions we raise in the book. Still, such knowledge—for instance, a stonecutter's knowledge of the required proportions and refinements of individual stones, and how to produce them—*is* a first step in that direction. Certainly, any account of the specific number of people who had access to these different kinds of knowledge would be purely speculative, at least within the scope of this project, but one's intuition that the fullest intellectual and philosophical engagement with the building would have inevitably involved a relatively small number of people (Pythagoreans or otherwise) seems right. All the same, the range of experiences that the building produces, from the most direct and unconscious to the most reflective and theoretical, seem crucially related.

Thus, understanding the Parthenon rightly is possible only when we appreciate the role of practical exposure to "problems in the arts" in a liberal education generally. This, we want to suggest, holds not only for this "institution of liberal education"; actually all institutions pursuing such a program of education have an interest in exposing the students in their care to such problems. Education in the liberal arts originated from a dialectical reflection on problems in the practical arts and more abstract thinking about them, as found in what we would now call "the exact sciences." Such education, in principle, ought always to be versed in such reflection. Or, more baldly still: humanities students ought to have enough quantitative competence to understand what questions in the exact sciences remain open and why they remain open. That, we believe, is the role of the problem-based study of mathematics in the Parthenon, and it is relevant as much for us today as for those who designed, built, and worshipped in the Parthenon.

Even if one is willing to accept our basic two-point hypothesis about the Parthenon as an institution of liberal education, and its corollary for liberal education more generally, though, there remains the following worry: what does the scholarly consensus tell us about the state of the art in Greek mathematical knowledge in the mid-fifth century, and does that consensus tell against our hypothesis? Is it really the case that much of what was known by the time Plato wrote the *Republic* (say 380 BCE) was in fact already known by a fair number of skilled artisans by the time the Parthenon was designed and built some sixty-five years earlier? Our reply begins by noticing that perhaps it was known but not demonstratively known, or put another way, known but not yet subject to deductive proof. This last concession is potentially decisive, as we hope to show.

Scholarly consensus holds that formalized proof like what we read in Euclid was entirely absent from Greek mathematics until at least the time of Plato's death. This does not, however, tell against our hypothesis that many of the most important findings first *demonstrated* in *Elements*—and crucial among these for us would be the propositions concerning mean proportionality, continuous proportion, and how these relate to square and cubic numbers—were in fact widely but perhaps not "demonstratively" or "formally" known by the middle of the fifth-century.[9]

Finally, even if our less-given-to-speculation colleague is convinced that our approach survives these two plausibility tests, there remains the following concern: such a well-developed community of practitioners of such knowledge would surely have produced some kind of traceable work that should inform us of who they were and what their research problems and possible solutions were. Why then, the objection would go, are we entirely without any documentation of these groups, of their participants' names and their findings? To this objection, we have two replies. First, the Parthenon itself is the primary source documentation of the community of researchers, whose research method was to work on the problem by designing the structure, and thus "publishing" their results not in a journal for specialists, but for everyone to experience in their civic, religious, and individual encounter with the building (which was a temple, built for the city as a whole with funding that the city secured from a mix of public and private sources). Second, while we believe that existence of one or more treatises having been written by the principal designers is entirely unnecessary to the argument, since the Parthenon speaks for itself, it does merit notice that Vitruvius refers to a book on the Parthenon by Iktinos and "Karpion" among a list of ancient architectural treatises, all now lost, at the beginning of Book VII of *De architectura*.[10] We will probably never know, but it is not impossible that this treatise was (or was in part) something like a guidebook to understanding the Parthenon as a site for solving problems in the interdisciplinary practice of mathematical arts. If this is so, then the formal analysis we will provide in part II might best be understood as akin to what this treatise would have presented. In short, we hope to show that the designers of the Parthenon saw their creation this way, and that they did so because of their vision of what we might call a liberal education.

2. The Parthenon and the Historiography of Greek Mathematics

If we are right in what we say elsewhere, then the relevant historiography needs to be amended at least to acknowledge that there was a *foundational* understanding of (at least) two mathematical objects generally thought to be understood in a reflective way for the first time in the mathematics of the first decades of the fourth century: continuous proportion (and its non-reducible-to-a-unit-or-multiples-of-a-unit correlate, *anthyphairesis*) and the relationship of the geometric and harmonic mean (a reflection that seems to arise from thinking about the double square and the issue of incommensurability).[11] Also, as we show in the work on the building itself (part II), its design entails a substantial theoretical reflection on the nature of the unit as constructed or discovered. Our suggestion in what follows amounts to this: if these features are present in the building, then it seems likely that Szabó (1978) was not mistaken in his central suggestion. We thus explore how it can be shown that more was known, in some sense on a theoretical level, at an earlier date than the standard story, offered by Knorr (1975), holds.

We very much share the view of Cuomo (2001) and Netz (1999, 2002) that the history of Greek and ancient mathematics cannot be pursued without constant back-and-forth attention to theoretical understanding and actual, practical use, as reflected in artefacts related to specific contexts, like the marketplace and the temple, the household, the state administration, and the library.[12] Drawing on this methodological perspective, our contribution to the historiography of Greek mathematics proceeds in three movements. First, we recall the general contours of the debate between Szabó (1978 [1969]) and Knorr (1975), both presenting a few of their key disagreements and discussing the rough consolidation of a position more or less like Knorr's. We will flag the somewhat provisional nature of this consolidation and the fact that all agree we could understand the pre-Euclidian period better—here following Cuomo (2001), Christianidis (2004), and Netz (2002, 2014). Next, we explore the possibility that both sides of the debate have an overly narrow view of the relationship between practical mathematical knowledge and its theoretization. This can be ameliorated, first, by attending to the central suggestion of Fowler (1999) regarding *logistikē* (and *anthyphairesis* in particular) in the theoretical mathematics of the early Academy and, second, by reopening the question of possible "legacies" of Near-Eastern mathematics in the *theory* as well as the *practice* of mathematical procedures. In the third and final movement of this section, we present an open-minded reading of the features that we elsewhere claim to be at the heart of the design of the Parthenon. By the end of this effort, we hope it will be accepted as *possible* that the Parthenon was designed by skilled artisans whose *theoretical* understanding of "cutting-edge mathematics" was great enough to have accomplished something that the present-day historiography of mathematics believes to have only been possible after the work of Theaetetus, Archytas and Eudoxus.

A return to the Szabó-Knorr debate is crucial for telling a story about the historical development of the theory of incommensurability, which itself seems central to pre-Euclidian mathematical knowledge. Two central components of Szabó's analysis are relevant. The first is his thesis that there was much more of a theoretical development in Greek mathematics during the fifth century than is acknowledged by the (then, and mostly now) conventional view that theoretically advanced mathematics began only in the early fourth century.[13] The second is his view of the chronological and doxagraphical priority of music theory; while all parties agree that the integration of work on number and work on geometry belongs to the early decades of the fourth century, Szabó (1978: 108–78) insists that this theoretical integration and advance was built on the basis of a prior theoretical achievement within music theory that developed out of musical practice and was already mature in the first half of the fifth century.

Through attention to the development of the three central mathematical disciplines (harmonics, geometry, and number), Szabó aims to show that even if nothing like the formal proof had developed in the mid-fifth century, by the time of Philolaus and Hippocrates there was already a rich tradition of truly theoretical mathematics. In particular, Szabó (1978: 14–33) argues that the crucial step in proving incommensurability dates to Hippocrates, whom he dates to having been active in Athens around 430 BCE.

This claim hinges on three subsidiary claims. First, according to Szabó (1978: 14), the construction of the mean proportional on a straight line—recorded as Proposition VI.13 in *Elements*—was already known by the time of Hippocrates. Second, the oldest demonstrative reasoning in proving mathematical truth is Epicharmus's theory of odd and even, dated "fairly accurately" (Szabó 1978: 25) to ca. 500 BCE. This is crucial because "this theory clearly culminated in the proof of the incommensurability of the diagonal and sides of a square." Third, provided sufficient attention is given to the integration of the three mathematical disciplines (Szabó 1978: 26–28), a chronology can be established (Szabó 1978: 28–29) that proceeds thus:

1. Musical theory of proportion, from which the terms of Eudoxan proportionality were borrowed. This first stage has two phases: (a) experiments with the monochord, giving rise to terminology for 2:1, 3:2, 4:3; (b) development, by

this means, of the technique of *anthyphairesis* (which Szabó translates as "successive subtraction").

2. Application of the "musical theory of proportions" to arithmetic (along the lines of what is formalized and recorded in *Elements*, Books 7–9).

3. Application of this proportion theory to geometry. This was done "of course" at "the time of the early Pythagoreans," working with the construction of the mean proportional.

4. Development of "mathematics within a deductive framework," here speaking of *Elements* and its concomitant theory of proof.

Szabó (1978: 29) stresses two points about this chronology. First, if things did proceed this way, then quadratic incommensurability "must have been known well before the time of Archytas." Second, "the discovery of incommensurability is due to a problem which arose originally in the theory of music." Of course, "knowing well" can mean a number of things, as critics of Szabó have pointed out. In proceeding, we suggest that "knowing well" be taken to mean that there was a deep theoretical engagement with this issue, embracing its relevance for all three emergent mathematical disciplines—geometry, arithmetic, and harmonics.

Szabó's chronology, and the "priority argument" in the development of a proof of incommensurability that it entails, involves Szabó (1978: 33–84) in a very extended argument to the effect that it is a mistake to attribute to Theodorus and to Theaetetus the decisive role in the proof of incommensurability in the period 410–370 BCE. Szabó's claim rests primarily on two grounds: (1) a philological argument focused on his understanding of the use of the term *dunamis* (as "square," "power," and so on) in mathematical texts from the earliest surviving fragments down to Euclid; (2) an interpretive argument concerning Plato's complicated intentions in associating the discovery of a *proof* of incommensurability with these two men in the dialogue that bears the younger man's name. Without entangling ourselves too much in these arguments—which have not proven persuasive[14]—let us briefly state the decisive moment in each of these arguments for Szabó. We will then discuss the basis of criticism thereof, and why we think we ought to consider this an open question.

Szabó (1978: 48) summarizes the philological argument this way: "Thus our previous conjecture to the effect that *dynamis* and *tetragonismos* originated at the same time inevitably leads to the conclusion that the creation of the concept of *dynamis* must have coincided with the discovery of how to *construct a mean proportional between any two line segments*." This is the crucial link for Szabó (1978: 50–54), insofar as he holds that Hippocrates (and not Archytas) originated the proofs about mean proportionality. In other words, Szabó believes the term "*dunamis*" began signifying "extent-in-square" at the same time that it was learned that one could *practically* construct a square by means of the mean proportional between two line segments. If this is true,[15] and if it is also true that Hippocrates was the first to publicize "proofs" (perhaps only informal proofs) concerning the mean proportional, then we can be confident that the crucial step in *proving* incommensurability in fact dates to around 430 BCE and not 410–370 BCE.

The second support for his chronology is the doubt cast on the centrality of Theodorus and Theaetetus. These arguments have not been well received, but we believe this has to do with hermeneutical questions involving the Platonic dialogue rather than any preponderance of historical evidence. We will re-present Szabó's three main points in the hopes of showing that if we keep an open mind about how to read a Platonic dialogue—and see, for instance, Gadamer (1980, 1986, 1991), Griswold (2002), and Nikulin (2010, 2012a) and our discussion of them in chapter 1, 1, on why we ought to—then we really ought to consider this matter very much open. We do not believe that Szabó "proves" that Theodorus and Theaetetus are not crucial for the development of a *proof* of the *theory* of incommensurability; we do believe, however, that he provides good reasons to doubt what reasons we have to accept their centrality in this story. First, Szabó (1978: 68–71) deploys a reading of the pun on "*dunamei dipous*" in *Statesman* 266a5–b1 to confirm that what Theaetetus describes in discussion with "young Socrates" (who also appears together with Theaetetus in *Theaetetus*) was common knowledge.[16] Second, Szabó (1978, 76) argues that the mistaken attribution of significance mostly derives from two sources, which he spends five pages trying to debunk: first, an ancient Scholium on *Elements* X.9; second, "a report which probably stems from Pappus' commentary on Book X and survives only in an Arabic translation."[17] Since both of these sources themselves rely mostly on a wrong-minded reading of the mathematics section (147c–148b[18]) of *Theaetetus*, Szabo (1978: 79) argues, there is no reason to see this attribution as anything other than a "false tradition." Finally, Szabó (1978: 79–84) presents a hermeneutical and philological analysis of the dialogue in the service of this answer: Plato presents Theaetetus as a very talented researcher, but also as young, naive, and overeager. The relevant section about the discovery of a theory of irrationals must be read in this light.

These arguments are surely not decisive, but neither are they decidedly wrong, we suggest. With them in mind, let us turn to Szabó's second major concern: the central importance and chronological priority of musical theory, which inspired an analytical program in number theory, which in turn inspires a reflection on the (im)possibility of fully integrating number theory and geometry—the latter having developed separately, without prior integration with the other two disciplines. Szabó's (1978: 108–78) account is that music theory arrives at the three means, and realizes there is no geometrical mean for the fourth, fifth, and octave. Then, approaching the double square, (Pythagorean) mathematicians discover mean proportionality between two lengths, whether numbers or not. Thus: music then arithmetic, then geometry.[19]

Knorr (1975: 9), unlike Szabó, believes that "by the time of Hippocrates both [the theory of congruence and the theory of similarity (based on proportion)] were already well developed," but that the combination of these two traditions (cf. Knorr [1975: 7]) began "only about fifty years later," with the contributions of Eudoxus. Knorr (1975: 49) insists that "evidence in the pre-Socratic literature discourages dating [the] discovery [of the theory of incommensurability] before ca. 430. One may recognize only after that time signs of a dialectical interest in the problem of incommensurability." The center of the debate, again, is in the interpretation of Plato's *Theaetetus*.[20] Knorr (1975: 116–7) argues against Szabó as follows. (1) Even if (the idea demonstrated in) Euclid VIII.18 is at work here, this only shows that "there is no *integer* which is the mean proportional between two terms that are not similar numbers," which is not enough to establish incommensurability. (2) Szabó's "view of the antiquity of the mathematics of the dialogue is an assumption," against which Knorr (1975: 82–86) has offered an argument, and which is not well supportable by the available documentary evidence. (3) Given Knorr's own view of the relative novelty of any theoretical understanding (and certainly proof) of incommensurability, it is "at least a reasonable counter-assumption that the number-theoretic foundation, upon which Theaetetus and his successors built their theory of incommensurability, had not yet achieved an advanced form at Theodorus' time." Knorr here underscores what Szabó himself acknowledges: that the relative novelty or antiquity of the discovery is a matter of speculation that—so far as extant documentary sources go now, as then—will never be definitively settled.

Given that the participants in the debate both acknowledge that definitive knowledge is impossible here, and that the matter is really about "who knew what when" within a fairly narrow research program in a relatively short time period, this might seem like a moot question—even more so because there is not a "world of difference" between these two positions. Nevertheless, determining which, if either, is correct would be very telling for the argument we make about the Parthenon, because if Knorr is right that the kind of integration Szabó sees in Hippocrates was only done in the time of Eudoxus, then it becomes difficult to impossible to believe that this integration, which we argue is integral to the design of the temple, would even have been thinkable, let alone doable, in the period 447–432 BCE.

So, we look for a way to advance this deadlock, and we turn to the role of *anthyphairesis*. Fowler (1999), asking different questions for different reasons, nevertheless focuses on just this knowledge procedure, and finds in it a suspicion of the "standard story" in the historiography of Greek mathematics. Fowler (1999: 4) notes that Knorr agrees with Szabó that "the theory of incommensurability will be perceived as contributing to important aspects of every part of the *Elements*, save for the oldest geometrical materials contained in Books I and III." They do so because they are both committed to "the standard story" about the history of Greek mathematics, which runs something like this: "The early Pythagoreans based their mathematics on commensurable magnitudes (or on rational numbers, or on common fractions *m/n*), but their discovery of the phenomenon of incommensurability (or the irrationality of √2) showed that this was inadequate. This provoked problems in the foundation of mathematics that were not resolved before the discovery of proportion theory that we find in Book V of Euclid's *Elements,*" while Fowler (1999: 4) "disagrees with everything in this line of interpretation."

For all that his account clearly owes to Knorr, Fowler's work actually serves as a reason to believe that the kind of sophistication that Szabó sees as present a few generations earlier than Knorr really is manifest—albeit with the crucial revision that it is not that "theoretical" advances occurred earlier than Knorr allows and as Szabó insists, but rather that mathematical *practice* was well "ahead" of its theorization and formalization, and thus that *anthyphairesis* and its insights were familiar to those working on the Parthenon in the middle of the fifth century. It may be that Fowler is entirely correct that much of what is traditionally viewed as "Pythagorean" mathematics had nothing to do with Pythagoras and his followers.[21] It may also be true that nothing like (1) the formal theory of proportion in *Elements*, V[22], or (2) the formal system of deductive proof presented in *Elements* as a whole[23] had been seriously developed during the fourth century, let alone the fifth. All the same, by underscoring the centrality of *anthyphairesis* as a *practice*—especially in addressing problems in geometry (Theaetetus), in music theory (Archytas), and in astronomy

(Eudoxus)—Fowler provides welcome corroboration of our suggestion that the practicing mathematicians and mathematically informed artisans of the generations working at the time of the Parthenon's construction used *anthyphairesis* as a means by which to test, geometrically, solutions to problems in harmonics. The "chronology debate" turns out to be intertwined with a much more fundamental debate about the very nature of Greek mathematics and its logical and methodological and ontological foundations. It becomes advisable, then, to reconsider the "foundations" of Greek mathematics. Christianidis (2004) provides the best overview of this issue, while Netz (1999, 2002) offers the most thoroughgoing analysis.[24]

Our own view is that a more solid understanding of Greek mathematics depends on a better understanding of its relation to Near-Eastern precedents. We wish to ask: What can we learn concerning the novelty or "revolutionary character" of classical Greek mathematics from the practice of *anthyphairesis* as it developed from the earliest (scantily) recorded sources of the sixth century through to its presentation in Euclid's *Elements*? In answering this question, it is worth noting that while much is controversial about the degree of novelty in Greek mathematics, no one denies that there are some indications of contacts between Greek mathematicians (especially astronomers) and Near-Eastern counterparts, certainly by the fifth century (Meton), maybe even earlier (Hesiod).[25] The best-established point of similarity is interest in dates of appearance of stars and constellation during the year, comparable to material in MUL.APIN, and already present in Greek sources of the eighth century.[26] Given this evidence of overlap of "research problems" and particularly of calculative techniques and knowledge procedures, we will suggest that the early theoretical advances in Greek mathematics are unlikely to have developed sui generis, but rather built on the work of Near-Eastern antecedents.

Netz (1999, 2002) and others are surely right that the formal-deductive framework of Greek mathematics beginning in the fourth century and proceeding therefrom is absolutely internal to the Greek tradition. All the same, we submit that both theoretical objects of mathematical knowledge and intricate mathematical procedures of great importance for the Parthenon were studied to a high degree of comprehension already by the mid-fifth century. If this is plausible, we further suggest that this is so not because of a burgeoning influence of "Greek-style" systematic, proof-theoretical knowledge procedures at that time, but rather through the reception of Near-Eastern antecedents. Specifically, interest in theoretical issues such as mean proportionality and periodicity and facility with practices such as *anthyphairesis* seem unlikely to have developed *after* the development of "Greek-style" mathematics that is largely a "fourth-century and later" phenomenon, as in Knorr (1975) and Bowen (1984). It is also relevant that scholars generally accept Proclus's account of Oenopides of Chios (fl. around and after 450 BCE) as the first to distinguish between theorems and problems, insist on geometry done only with a compass and a straight edge, and draw a perpendicular straight line from a given point to a given straight line. If this is true, it certainly signals a functioning and fairly mature context of theoretical mathematical knowledge production *in Athens*—that is, knowledge pursued solely for its own interest and without relation to physical and practical production—by the time of the design of the Parthenon.[27]

It is impossible to avoid controversy in pointing in this direction[28] because the possibility of Near-Eastern influences on Greek mathematics of the Euclid type is very difficult to establish, partly because of the chronology (most of the evidence for Babylonian mathematics is early second millennium BC) and partly because the Greek deductive-demonstrative style of mathematics seems very remote from the algorithmic problem-oriented style of the Babylonian texts. The current orthodoxy is that Babylonian mathematics is not closely relevant to the development of Euclidean mathematics. Specifically, this "orthodox account" holds that the "rediscovery" of the putatively unique and transformative nature of the progress in mathematical knowledge in and around Plato's Academy during his lifetime and the century after, was integral to the develop of "mathematics as we know it," which essentially began in this period. Integral to the development of this account in the nineteenth century—and those works that followed through the middle decades of the last century—was the conviction that while practical mathematical understanding was developed much earlier and to a much greater extent in other places (especially Mesopotamia and Egypt) than in classical Greece, it was the Greeks alone who sought to develop a proper "theoretical" understanding of these mathematical objects.

The orthodoxy, though, has been challenged, most directly and fully by Friburg (2007). As he notes in his preface, in searching for connections between Babylonian and Greek mathematics, he was compelled to offer very new interpretations of some of the thorniest issues in the debates internal to the historiography of Greek mathematics. Among other things, he comes to propose a new understanding of Book 2 of *Elements* that would (if true) resolve the debate about "geometrical algebra" by showing that what is at work in these propositions is not geometrical algebra, but an "abstract, non-metric reformulation" of "systems of equations in Babylonian metric algebra."[29] Similarly revisionist arguments are made with respect to some of the problems relevant for our project, such as Book

10 propositions on irrationals (addressed by Friburg in chapters 3–5), and possible Babylonian sources for Hippocrates's work (chapter 12)—which is especially interesting for a possible full rehabilitation of Szabó's argument concerning "Hippocrates v. Archytas" in the first elaboration of mean proportionality. Like Friburg, we would suggest that this approach seems reasonable since grounds for rejecting influence seem to us (as to him) more based on prejudice than reason.

Philolaus (ca. 475–ca. 385 BCE[30]) is a central figure for the questions raised in this section, and thus makes for an excellent case study in attempting to address them. First, his intellectual career closely coincides with, if largely slightly postdates, the design of the Parthenon, and shows a concern for precisely the kind of integration of the mathematical disciplines organized thematically around "harmony" that we find there. Moreover, while we will primarily focus on another discovery or invention that is attributed to him, the "Pythagorean Tuning" that is classically preserved in (what Huffman [1993] refers to as) Fragments 6/6A of Philolaus's *On Nature* is integral to the mathematics of the Parthenon, as we show elsewhere. Finally, his work carries a distinct trace of what Babylonian mathematics did best and the field in which the best evidence of contact between Greek and Near-Eastern mathematics has been located: astral science—specifically, in the theory of the moon, the theory of the sun, and the position of the ecliptic.

Take the case of Philolaus's "great year." Huffman (1993: 276–79) provides a thorough account of what is known about this intellectual achievement, which purports to name the period within which solar years coincide with lunar months: namely, 59 solar years (where each solar year has a value of 364½ days), or 729 lunar months (where each lunar month has a value of 29½ days). As Huffman (1993: 277) pointedly says: "The crucial question is how did Philolaus arrive at this set of numbers?" Huffman considers two main alternatives, one in which Philolaus is adopting and adapting the value that Oenopides (mentioned above, and a slightly older contemporary of Philolaus) had arrived at: 730, and the other in which he derives it directly from recorded observations. If he revised an earlier value, the supposition goes, this could be either because of some preference in working with the observations—like the value it gives for the solar year (of 364½ days, rather than $365^{22}/_{55}$ days—or because of the inherent attractiveness of 729 as the square of the cube of 3.[31]

This all seems quite right as far as it goes. But it also seems clear that the central motivation of the entire enterprise that would lead you to posit a "great year"—which Huffman (1993: 276) identifies as "an attempt to harmonize two important ways of measuring time, the lunar month and the solar year"—is not sui generis within Philolaus's cosmology, or within Pythagoreanism. Both the observational data and the idealized values with which Philolaus and his Greek contemporaries were working came to them from Egypt and Mesopotamia. Given this fact, Huffman's analysis leaves aside a possibility worth serious consideration. Namely, is it not possible that solar year and lunar month periodicity presses itself on these fifth-century Greek sources through the determination of the same issue in the Babylonian astronomical work of the seventh and sixth centuries?

While it is difficult to establish connections for the reasons stated above, here is an instance where we have Babylonian source materials—specifically "goal-year tablets," which make predictions of where certain heavenly bodies will be at certain times in a given year, and "lunar prediction tablets," which focus on the moon over long periods—from the time in question. Since the 1990s, a great deal of work[32] has been done on exemplars of this tradition that have been definitively dated to the period (e.g., tablets from ca. 642–640 BCE, 593 BCE, 523 BCE) and others whose dating is not precisely known but date from some time not earlier than the fifth century BCE and not later than the third century BCE. These texts display two features that relate directly to the question Huffman raises concerning Philolaus's process in arriving at his value for the Great Year.

The first is an interest in what Huffman (1993) calls, with respect to Philolaus, the "harmonization" of the solar year and the lunar month. In the case of Mesopotamian astral science,[33] there was a long tradition of interest in this. While the underlying motivation of the Near-Eastern precedent displays a significant difference from what we can glean as being Philolaus's motivation, the observational data with which Philolaus and his contemporaries could seek out significant number patterns is surely owed to this earlier tradition. But Greek astronomers and mathematicians probably received a good deal more than just the data. For instance, bearing in mind the work of Huber and Steele (2007) on the so-called "Saros function"—an eighteen-year solar cycle—that is transmitted in lunar prediction tables dated to 642–640 BCE, and also the work of Britton (2002) on a lunar prediction table (dated to ca. 620 BCE) that uses a twenty-seven-year solar cycle, one detects a strong consonance between Philolaus's interest in powers of three and the repeated thematic and methodological use of multiples (and especially powers) of three in these Near-Eastern antecedents of which Philolaus or those with whom he worked might well have been aware.

The second noticeable similarity in the research projects of Philolaus and Near-Eastern astronomy of the seventh through fifth centuries is the direct interpretation of data as itself an object of observation. This appears to have been the hallmark of Babylonian astral science in particular: already by the seventh century, we see the development of what is called the "linear zigzag function," which describes the pattern that emerges on a table inscribed on these tablets, given the values for mean speed (for moon, sun, or both) through the phases of the moon over a period of solar years. From especially the fifth century on, this sort of function is used to give sequences of numbers tabulated for equidistant intervals of time, from which periods could be calculated, and eclipses could be predicted. This back-and-forth procedure from tabular data to worldly phenomena became integral to Greek mathematics, especially astronomy. For instance, as Evans (1998) relates, Hypsicles (in a lost astronomical work) used Greek mathematical rhetoric of a Euclidian kind to present precisely Babylonian values of the "linear zigzag function" for the mean speed of the sun and the moon. We do not have a record of precisely when Greek mathematicians began working with the periodic functions of Babylonian astronomy. But given the contacts we know to have been established between the seventh and fifth centuries, there is no reason to believe that Hypsicles or his contemporaries were receiving the zigzag function for the first time. In any case, the interest in periodicity, and the habit of noticing periodicity in data directly, and *then* bringing it to objects of observation or construction is shared by the Near-Eastern astral science of the seventh century and Philolaus.

This brief investigation of how Philolaus's "Great Year" calculation can at best provide a test case for both the plausibility of a development in fifth-century Greek mathematics that is continuous with Near-Eastern predecessors and the interpretive possibilities this allows within the framework of contextualizing the (very scantily recorded) Greek mathematics of this period; it does not, we know, *prove* anything. It does suggest that the current, and continuing, inquiry about the development of Greek mathematics at the time of the design of the Parthenon could benefit from a more extended comparative analysis of the kind we have initiated here. What's more, it should at least be clear that the kind of knowledge procedures employed in the "algorithmic problem–oriented" style of Babylonian mathematics of these centuries is significantly similar to Greek mathematical procedures prior to the formalization and introduction of deductive proof in the fourth century.

Noticing the relevance of a problem-based, "trial and error" approach to theoretical mathematics that links the work of Philolaus to mathematical practice in the seventh- and sixth-century BCE Near-Eastern astral science provides insight into how mathematical knowledge procedures functioned in the Parthenon. We see in the temple's design precisely the kind of "algorithmic problem–oriented style" we know to be the hallmark of Near-Eastern approaches. Analyzing the mathematically informed features of the Parthenon design, it seems quite likely that the *process* of coming to these design features is through the *reflective* deployment of a variety of instruments derived from "algorithmic problem–oriented style" mathematical practice. If that was happening in Athens at the time of the temple's design, it seems natural to ask: How did they know how to do this? Why did they choose to do so thus? We conclude this introduction with a first approximation of an answer to this question, which serves as the basis for the sustained analysis offered in part II.

On the hypothesis (1) that the mathematical features we find in the Parthenon are really there, and (2) the central findings of this introduction so far are plausible, we now close with an account of why the mathematical features in question might have been introduced to the design of the Parthenon. We begin by recalling the "Plato and the mathematicians" discussion,[34] in which the use of the difference in methodology between one kind of mathematics and another is key for understanding Plato's critique of the (Greek) mathematicians (of his time) in *Republic* 6 and 7 (510c–d, 528b–d, 529b–c, 529e–30b, 531b–c).[35] What light is shed on these conversations by the findings of the second section of this introduction concerning "Babylonian-style" math and "Greek-style" math? Plato argues that by simply applying their procedures, which they treat as granted setting-stones (hypotheses), without questioning their principles, they are guilty of doing least what mathematics has the greatest possibility to achieve: leading us to the forms. Interpreters will probably continue to debate the exact nature of Plato's critique and what it means about his own mathematical understanding and the actual practice of the mathematicians of his time, and we offer a fuller treatment of this in chapter 3. Here, we simply note that the determination of what exactly the mathematical practice is that Plato's Socrates critiques in Book 7 of the *Republic* shines a light on the question of possible continuities between the mathematicians Plato is criticizing and the Near-Eastern mathematicians that the second section has tried to show at least might have had an influence on them.

Further contextual light is thrown on this matter by thinking through Proclus's account of the debate between what we can call the "constructivist" and the "realist" philosophies of mathematics within the Academy. By teasing out the background of these positions with the epistemology and ontology of mathematical objects, we can see how the constructivist approach maps onto the practical-procedural approach we have explored with respect to the Near-Eastern precedents of early Greek mathematics, while the realist approach relates to the development of formal-deductive procedures that decisively break with such practices. Nikulin's (2012b) work on "indivisible lines" and Negrepontis's work on periodic *anthyphairesis* draw this in deeper relief. What emerges here is that key features of periodic *anthyphairesis* as described by Negrepontis (2012) correlate strongly with the design of the Parthenon as a mathematical construction. This vindicates Negrepontis's explanation—itself an echo of Knorr (1975) and Szabó (1978), as reconstructed by Fowler (1999)—of Aristotle's claim from *Topics* (158b22f.) about a pre-Eudoxan approach to proportions in Greek mathematics through finite and infinite *anthyphairesis*.[36] This itself points toward the possibility of a robust, *theoretical* reflection on recursive procedures, like those used in the construction of the Parthenon, having already existed by the time of the Parthenon's design. Such a reflection would have addressed, for instance, the question of why some recursive expansions in square yield continuous proportions of whole numbers, while others arrive at the "infinite *anthyphairesis*" Plato is worried about in dialogues such as *Theaetetus*, *Philebus*, and *Statesman*.

A synoptic rehearsal of three crucial elements of the temple's design described above demonstrates this.[37] First is periodicity itself. As we saw in comparing the dimensions of the Parthenon's stylobate with standard Doric intercolumniation, a 5:1 (80:16) ratio of intercolumniation to triglyph, the Parthenon's continuous proportions give a ratio of 81:16 between these elements. The refinements involved with fitting these two slightly different things together is a first instance of the problem of *harmonia* in the Parthenon. Second, the construction of the unit. As described above, using this particular continuous proportion, at various scales, allowed for the building's overall cubic proportions, in the continuous proportion of 81 (length) : 36 (width) : 16 (height) to have as its unit the real, visible triglyph module. Last, the harmony of the whole. As in the construction of the ancient Greek musical scale, and as Nikulin (2012b) details in his discussion of the divided line and its importance to Plato, so too in the building of the Parthenon, the consideration of a geometric object in arithmetical terms as a means of forging an aesthetic (and ontological) whole gave rise to an irreducible tension between magnitude and multitude. This tension is made productive in all three cases as the decisive factor in the thinking-through of harmonics (*harmonia*), that is, in the joining together of conflicting elements to create a unity. In Doric architecture, this problem emerges most fully in the need for a harmonious articulation of the building's corner, an issue engaged with unique intensity in the Parthenon, via a unique (to our knowledge) approach to a "distribution of the difference" problem.

However much might remain undecidable in what we have presented, it should be clear why the interpretation of the mathematical features of the Parthenon opens up large questions about the development of Greek mathematics in the first decades of the second half of the fifth century. The advanced state of *practical* mathematics in mid-fifth-century Athens rekindles the old but not extinguished flame in debates concerning the sui generis nature of theoretical mathematics in fourth-century Greece and the possibility that there was real continuity between Greek mathematicians and their Near-Eastern predecessors. This suggests that we would do well to develop a deeper appreciation of the "algorithmic problem–oriented style" of many pre-Euclidian Greek mathematicians—at least, we would maintain, of those knowledge practitioners involved in the design of this building, and also for those (in both the western colonies and on the mainland) with whom they were obviously in productive contact. This finding seems consistent with Netz's (2002) account of Greek "counter culture" as a *practical* affair in the "cognitive history" of the sixth and fifth centuries. But if there is a conscious attempt to introduce this mathematics into this building in a thematized and programmatic way, what is the intellectual background of that attempt? We attempt an answer by investigating the state of the art in Greek mathematics during Plato's long intellectual career in part I; we then move on in part II to critically reconstruct the work the Parthenon was doing as a "vanishing mediator" between the seventh- and sixth-century Near-Eastern mathematics we have seen exemplified in the solar and lunar tables above and the work of Plato and his contemporaries in and around the early Academy.